

Stochastic approach to inflation. II. Classicality, coarse graining, and noises

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In this work we generalize a previously developed semiclassical approach to inflation, devoted to the analysis of the effective dynamics of coarse-grained fields, which are essential to the stochastic approach to inflation. We consider general non-trivial momentum distributions when defining these fields. The use of smooth cutoffs in momentum space avoids highly singular quantum noise correlations and allows us to consider the whole quantum noise sector when analyzing the conditions for the validity of an effective classical dynamical description of the coarse-grained field. We show that the weighting of modes has physical consequences, and thus cannot be considered as a mere mathematical artifact. In particular we discuss the exponential inflationary scenario and show that colored noises appear with cutoff dependent amplitudes. [S0556-2821(99)05206-6]

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I. INTRODUCTION

Inflation solves several difficulties that arise in the very early universe, such as the horizon, flatness and monopole problems, and in addition to this it provides a mechanism for the creation of primordial density fluctuations needed to explain the structures which are now present in the Universe [1,2]. The most widely accepted approach assumes that the inflationary stage is driven by a quantum scalar field φ with a potential $V(\varphi)$. Within this perspective, stochastic inflation seeks to describe the dynamics of this quantum field on the basis of a splitting of φ in a homogeneous and an inhomogeneous component. Usually the homogeneous one is interpreted as a classical field that arises from a coarse-grained average over a volume larger than the observable universe, and plays the role of a global order parameter [3]. All information on scales smaller than this volume, such as density fluctuations, is contained in the inhomogeneous component. Although this theory is widely used and accepted as a general framework, it presents inconsistencies and has been subjected to several criticisms [4–6]. Its main problems are related to the treatment of the global order parameter as a classical field, and the description of the quantum fluctuations as classical ones.

In a previous work [7] we assumed that the coarse-graining volume is defined so that it leads to a Heaviside function in momentum space. This is a choice that simplifies the mathematical development because it leads to a noise with a very simple white spectrum, but at the same time with singular correlations. A usual assertion is that the dynamics for the coarse-grained field is not very sensitive to this choice, but its actual implications on the resulting dynamics are not well understood. The present paper is mainly devoted to the study of this last point, together with an analysis of classicality conditions (in our case commutativity condi-

tions). To do this we develop a general formalism for coarse-graining volumes with arbitrary shape in momentum space, and analyze the consequences of this shape on the emergence and structure of the classical effective regime. In addition to this there is another improvement respect to our previous work. There we stated a sufficient condition for a classical description, neglecting the contribution of a sector of the noise [9]. Here we analyze the whole quantum noise sector, and thus the characterization of our classicality criterion is complete. This analysis enhances our understanding of the role of the shape of the coarse-graining in the effective classical dynamics, and also provides us with a well defined regularization scheme to treat the usual sharp cutoff in momentum space.

In the work just mentioned [7] we analyze the emergence of a classical behavior of the order parameter on the basis of a semiclassical approach. The inflaton field Lagrangian is

$$\begin{aligned}\mathcal{L}(\varphi, \varphi_{,\mu}) &= -\sqrt{-g} \left[\frac{1}{2} (g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu}) + V(\varphi) \right] \\ &= a^3 \left(\frac{1}{2} \dot{\varphi}^2 - \frac{1}{2a^2} (\nabla \varphi)^2 - V(\varphi) \right),\end{aligned}\quad (1)$$

for a Friedmann-Robertson-Walker (FRW) metric, $ds^2 = -dt^2 + a(t)^2 d\vec{r}^2$. The equation of motion that results for the scalar field operator is

$$\ddot{\varphi} - \frac{1}{a^2} \nabla^2 \varphi + 3H\dot{\varphi} + V'(\varphi) = 0, \quad (2)$$

where the overdot represents the time derivative and $V'(\varphi) = dV/d\varphi$, and the metric, given by the Hubble parameter $H = \dot{a}/a$, evolves according to

$$H^2 = \frac{4\pi}{3M_p^2} \left\langle \dot{\phi}^2 + \frac{1}{a^2} (\vec{\nabla} \phi)^2 + 2V(\phi) \right\rangle. \quad (3)$$

We decompose the scalar field in its mean value, which by assumption satisfies a classical equation of motion, plus the quantum fluctuations, $\phi = \phi_{cl} + \phi$ with $\langle \phi \rangle = 0$, up to linear terms in ϕ . In such a way the equations of motion reduce to a set of two classical equations which give the evolution of the field ϕ_{cl} and the Hubble parameter. To be consistent with the FRW metrics, we assume that ϕ_{cl} is a homogeneous field, and thus we have the following classical equations:

$$\ddot{\phi}_{cl} + 3H\dot{\phi}_{cl} + V'(\phi_{cl}) = 0, \quad (4)$$

$$H^2 = \frac{8\pi^2}{3M_p^2} \rho, \quad (5)$$

where ρ is the energy density, $\rho = \frac{1}{2}\dot{\phi}_{cl}^2 + V(\phi_{cl})$, and one operatorial equation for the quantum fluctuations:

$$\ddot{\phi} - \frac{1}{a^2} \nabla^2 \phi + 3H\dot{\phi} + V''(\phi_{cl})\phi = 0. \quad (6)$$

In this last equation H and V''_{cl} are functions of t given by Eqs. (4), (5). In this context we developed the analysis of the emergence of a classical regime for the inflationary dynamics [7].

The characteristic time scale for the inflaton field can be defined by $\tau_d = \phi_{cl}/\dot{\phi}_{cl}$, and hence its relation with the Hubble time scale $\tau_H = H^{-1}$ is given by

$$\vartheta \equiv \frac{\tau_d}{\tau_H} = \sqrt{\frac{2}{3}} \frac{2\pi}{M_p} \frac{\phi_{cl}}{\dot{\phi}_{cl}} \rho^{1/2}, \quad (7)$$

and the number of e-folds in a given period of time is

$$N_c = \int_{t_0}^{t_0 + \delta t} dt H = \int_{\phi_0}^{\phi_{cl}} d\phi_{cl} \frac{\vartheta}{\phi'_{cl}}. \quad (8)$$

If we are interested in an exponential inflationary period, i.e. when the slow roll of the field holds, then the conditions $\Theta = (M_p^2/4\pi)(H'/H)^2 \ll 1$ and $(M_p^2/4\pi)(H''/H) \ll 1$ must be satisfied [8]. The end of inflation, when the scale factor stops accelerating, is given precisely by $\Theta(\phi_{cl}) = 1$, which determines ϕ_{cl}^{end} . At this point we have $\dot{\phi}_{cl}^{end} \simeq -V'(\phi_{cl})/3H$ and $H^2 = (8\pi^2/3M_p^2)V(\phi_{cl})$, so that

$$\vartheta \sim -\frac{8\pi^2 \phi_{cl}}{M_p^2} \frac{V(\phi_{cl})}{V'(\phi_{cl})}, \quad (9)$$

and

$$N_c = -\frac{8\pi^2}{M_p^2} \int_{\phi_0}^{\phi_{cl}} d\phi'_{cl} \frac{V(\phi'_{cl})}{V'(\phi'_{cl})}. \quad (10)$$

A solution to the horizon problem requires $N_c \gtrsim 60$, and this in general implies that $\tau_d > \tau_H$.

The study of the quantum component becomes much simpler if we redefine the ϕ field such that the equation of motion (6) does not have a first order term, $\phi = e^{-(3/2)\int dt H} \chi$. The equation of motion for the field operator χ is

$$\ddot{\chi} - \frac{1}{a^2} \nabla^2 \chi - \frac{k_0^2}{a^2} \chi = 0, \quad (11)$$

where $k_0^2 = a^2(\frac{3}{4}H^2 + \frac{3}{2}\dot{H} - V'')$. Thus χ can be interpreted as a free scalar field with a time dependent mass parameter. It can be expanded in a set of modes $\xi_k(t)e^{i\vec{k}\cdot\vec{r}}$:

$$\chi(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int d^3k [a_k \xi_k(t) e^{i\vec{k}\cdot\vec{r}} + \text{H.c.}], \quad (12)$$

where the annihilation and creation operators satisfy the usual commutation relations for bosons:

$$[a_k, a_{k'}^\dagger] = \delta(\vec{k} - \vec{k}'), \quad [a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0, \quad (13)$$

while the modes are defined for the equation of motion

$$\ddot{\xi}_k + \omega_k^2 \xi_k = 0, \quad (14)$$

with $\omega_k^2 = a^{-2}(k^2 - k_0^2)$. The function $k_0^2(t)$ gives the threshold between an unstable infrared sector ($k^2 < k_0^2$), which includes only long wavelengths relative to the coarse-graining scale, and a stable short wavelength sector ($k^2 > k_0^2$). We adopt the normalization condition $\xi_k \dot{\xi}_k^* - \dot{\xi}_k \xi_k^* = i$ for the modes, such that the field operators χ and $\dot{\chi}$ satisfy canonical commutation relations.

In the section below we introduce a weight function to define the coarse-grained field and develop the general formalism, showing the importance of an adequate definition of the quantum noises and the conditions to properly derive a classical regime. After that, in Sec. III, we apply this approach to the inflationary inflation scenario and show the need for considering the proposed noise definition to prove the emergence of a classical regime. The last section is devoted to some concluding remarks.

II. THE GENERAL APPROACH

In order to define the coarse-grained field, an effective field which is an average of the long wavelength modes, we use a weight function. We assume that this is an isotropic function which contains only one parameter b with length dimensions, so that it is of the form $b^{-3}g(r/b)$, which is always larger than the causal horizon. The coarse-grained field χ_b is defined by

$$\begin{aligned} \chi_b &\equiv \frac{1}{(2\pi)^{3/2}} \frac{1}{b^3} \int d^3r g(r/b) \chi(\vec{r}, t) \\ &= \frac{1}{(2\pi)^{3/2}} \int d^3k G(\vec{k}) [a_k \xi_k(t) + \text{H.c.}], \end{aligned} \quad (15)$$

with

$$G(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3r}{b^3} e^{i\vec{k}\cdot\vec{r}} g(r/b) \\ = \sqrt{\frac{2}{\pi}} \frac{1}{kb^3} \int dr r g(r/b) \sin kr, \quad (16)$$

or, reciprocally,

$$g(r/b) = \sqrt{\frac{2}{\pi}} \frac{b^3}{r} \int dk k \sin kr G(k). \quad (17)$$

These equations can be written in terms of the dimensionless variables $\beta = kb$ and $\rho = r/b$ as follows:

$$G(\beta) = \sqrt{\frac{2}{\pi}} \frac{1}{\beta} \int d\rho \rho \sin \kappa \rho g(\rho), \quad (18)$$

$$g(\rho) = \sqrt{\frac{2}{\pi}} \frac{1}{\rho} \int d\beta \beta \sin \beta \rho G(\beta), \quad (19)$$

from Eq. (15), we obtain for the derivatives of χ_b :

$$\dot{\chi}_b = \frac{1}{(2\pi)^{3/2}} \int d^3k [\dot{G}(\beta) a_k \xi_k(t) + G(\beta) a_k \dot{\xi}_k(t) + \text{H.c.}], \quad (20)$$

$$\ddot{\chi}_b = \frac{1}{(2\pi)^{3/2}} \int d^3k \{ a_k [(\ddot{G}(\beta) - \omega_k^2 g_b) \xi_k + 2\dot{G}(\beta) \dot{\xi}_k] \\ + \text{H.c.} \}. \quad (21)$$

The parameter b is chosen so that the k^2 term, that is, the noise independent of the weight function, can be neglected in the equation of motion for χ_b . The appropriate value can be obtained from the equation of motion for the modes, Eq. (14). We can choose the characteristic length of the distribution larger than the horizon scale, i.e. $b^{-1} = \varepsilon k_0$ with $\varepsilon \ll 1$. From the definition of χ_b and the expression for $\ddot{\chi}_b$, assuming that we are only considering infrared modes with $k^2 \ll k_0^2$, we obtain the equation of motion for the coarse-grained field [9]:

$$\ddot{\chi}_b - \frac{k_0^2}{a^2} \chi_b = \eta_b + \kappa_b, \quad (22)$$

where

$$\eta_b = \frac{1}{(2\pi)^{3/2}} \int d^3k [a_k \ddot{G}(\beta) \xi_k + \text{H.c.}], \quad (23)$$

$$\kappa_b = \frac{2}{(2\pi)^{3/2}} \int d^3k [a_k \dot{G}(\beta) \dot{\xi}_k + \text{H.c.}]. \quad (24)$$

The dependence of the function $G(\beta)$ on t is given only through the parameter b , and thus, using the relation $\partial_b G = (k/b) \partial_k G$, we can write

$$\dot{G}(\beta) = \frac{\dot{b}\beta}{b} \partial_\beta G(\beta), \quad (25)$$

$$\ddot{G}(\beta) = \left(\frac{\dot{b}\beta}{b} \right) \partial_\beta G(\beta) + \left(\frac{\dot{b}\beta}{b} \right)^2 \partial_\beta^2 G(\beta). \quad (26)$$

Taking into account the algebra for the creation and annihilation operators, the operators χ_b , η_b and κ_b satisfy the commutation relations

$$[\chi_b(t), \eta_b(t')] = 0, \quad (27)$$

$$[\chi_b(t), \kappa_b(t')] = \frac{2}{(2\pi)^3} \int d^3k G(\beta) \dot{G}(\beta') (\xi_k(t) \dot{\xi}_k^*(t') \\ - \dot{\xi}_k^*(t) \xi_k(t')), \quad (28)$$

$$[\eta_b(t), \kappa_b(t')] = \frac{2}{(2\pi)^3} \int d^3k \ddot{G}(\beta) \dot{G}(\beta') \\ \times (\xi_k(t) \dot{\xi}_k^*(t') - \dot{\xi}_k^*(t) \xi_k(t')), \quad (29)$$

and the quantum noises have the correlation functions

$$\langle \kappa_b(t) \kappa_b(t') \rangle = \frac{4}{(2\pi)^3} \int d^3k \dot{G}(\beta) \dot{G}(\beta') (\dot{\xi}_k(t) \dot{\xi}_k^*(t')), \quad (30)$$

$$\langle \eta_b(t) \eta_b(t') \rangle = \frac{1}{(2\pi)^3} \int d^3k \ddot{G}(\beta) \ddot{G}(\beta') (\xi_k(t) \xi_k^*(t')), \quad (31)$$

$$\langle \kappa_b(t) \eta_b(t') \rangle = \frac{2}{(2\pi)^3} \int d^3k \dot{G}(\beta) \ddot{G}(\beta') (\dot{\xi}_k(t) \xi_k^*(t')), \quad (32)$$

with $\beta = kb(t)$ and $\beta' = kb(t')$. The quantum character of the fields becomes apparent through the non-null commutation relations and the complex correlation functions of the noises. To have an effective classical theory it is necessary that the non-null commutators be irrelevant, and consequently that the correlations be real.

To simplify the discussion we will assume that $G(\beta)$ is non null only in the range $0 < \beta \leq \bar{\beta}$, and that its derivatives are practically null for all values of β , except in a domain $(\bar{\beta} - \Delta\beta/2, \bar{\beta} + \Delta\beta/2)$, with $\Delta\beta \ll \bar{\beta}$, where the modes ξ_k vary slowly. This last interval corresponds to the “wall” of the coarse-graining domain, which we consider to be relatively well defined. Under these assumptions Eqs. (27)–(29) at $t = t'$ can be approximately written:

$$[\chi_b(t), \kappa_b(t)] \simeq -\frac{3}{\pi^2} \frac{\dot{b}\bar{\beta}^3}{b^4} G(\bar{\beta}) \text{Im}(\xi_k \dot{\xi}_k^*)|_{k=\bar{\beta}/b}, \quad (33)$$

$$[\eta_b(t), \kappa_b(t)] \simeq \frac{4}{\pi^2} \frac{\dot{b}\bar{\beta}^4}{b^5} \int d\beta (\partial_\beta G)^2 \text{Im}(\xi_k \dot{\xi}_k^*)|_{k=\bar{\beta}/b}, \quad (34)$$

where the b , ξ_k and G functions are evaluated at t . Here it becomes evident that the main difficulty in considering the equations of motion as classical is the κ operator. It does not commute with χ and η , whereas the latter ones do between them. This drawback can be overcome only if the contribution of κ is negligible compared with that of η . The contributions of the different terms can be weighted by their rms values. Assuming that the assumptions discussed before hold, we have

$$\langle \kappa_b(t) \kappa_b(t) \rangle = |\dot{\xi}_k|^2|_{k=\bar{\beta}/b} I_1, \quad (35)$$

$$\langle \eta_b(t) \eta_b(t) \rangle = |\xi_k|^2|_{k=\bar{\beta}/b} I_2. \quad (36)$$

Thus a necessary condition for classicality is

$$\left| \frac{\langle \kappa_b(t) \kappa_b(t) \rangle}{\langle \eta_b(t) \eta_b(t) \rangle} \right| \sim \left| \frac{|\dot{\xi}_k(t)|^2}{|\xi_k(t)|^2} \right|_{k=\bar{\beta}/b} \left| \frac{I_1}{I_2} \right| \ll 1, \quad (37)$$

where I_1 and I_2 are given by

$$I_1 \simeq \frac{2}{\pi^2} \frac{b^2 \bar{\beta}^4}{b^5} \int d\beta (\partial_\beta G)^2, \quad (38)$$

$$I_2 \simeq \left(\frac{b}{2\bar{\beta}} \right)^2 I_1 + \frac{b^4 \bar{\beta}^6}{b^7} \int d\beta (\partial_\beta^2 G)^2, \quad (39)$$

and hence

$$\frac{I_2}{I_1} \simeq \frac{1}{4} \left(\frac{b}{\bar{\beta}} \right)^2 \left(\left(\frac{b\bar{\beta}}{b^2} \right)^2 + 2\pi^2 \bar{\beta}^2 \frac{\int d\beta (\partial_\beta^2 G)^2}{\int d\beta (\partial_\beta G)^2} \right) \gtrsim \frac{1}{4} \left(\frac{b}{\bar{\beta}} \right)^2. \quad (40)$$

From here we can state a necessary condition

$$\left| \frac{\dot{\xi}_k(t)}{\xi_k(t)} \right|_{k=\bar{\beta}/b} \ll \frac{b}{\bar{\beta}}, \quad (41)$$

for relation (37) to hold. Given the coarse-grained field defined by $G(\beta)$ and the character of inflation given by H , this relation states a condition to be satisfied by the modes in the threshold sector between the unstable infrared sector and the stable short wavelength sector ($k^2 > k_0^2$), given by $k_0 = 1/\varepsilon b$, so that the coarse-grained field admits a classical description.

To have a classical regime there is another condition to be satisfied, namely the correlation function of $\eta(t)$ must be real. In general the correlation functions decrease rapidly with $(t-t')$, and hence we can approximate $t' \simeq t + \delta t$, and take the leading order term using $\xi_k(t') \simeq \xi_k(t) + \dot{\xi}_k(t) \delta t$. To this time increment corresponds a β variation $\delta\beta \sim (\beta \dot{b}/b) \delta t$. Up to linear contributions in δt the η correlation function is

$$\langle \eta_b(t) \eta_b(t') \rangle = \frac{1}{(2\pi)^3} \int d^3 k \ddot{G}(\beta) \ddot{G}(\beta + \delta\beta) |\xi_k|^2 \times \left(1 + \frac{\dot{\xi}_k^*}{\xi_k^*} \delta t \right). \quad (42)$$

Hence the condition to have a real η correlation function becomes

$$\text{Im} \left. \frac{\dot{\xi}_k}{\xi_k} \right|_{k=\bar{\beta}/b} \ll \delta t^{-1} \simeq \frac{\beta}{\delta\beta} \frac{\dot{b}}{b} \ll \frac{\dot{b}}{b}, \quad (43)$$

but if Eq. (41) is satisfied this last relation is also satisfied. Therefore, in general, when η dominates its time correlations are practically real.

In fact, these conditions are not sufficient to ensure that we have an effective classical dynamics because, although they warrant that the correlation $\langle \kappa_b(t) \kappa_b(t') \rangle$ is negligible with respect to $\langle \eta_b(t) \eta_b(t') \rangle$ and also that this last correlation function can be considered as a real one, they are not enough to warrant that $\langle \kappa_b \eta_b + \eta_b \kappa_b \rangle$ is simultaneously negligible.

As was already pointed out in our previous article, the noises appear only in the form of the right hand side term of Eq. (22), so that its decomposition in terms of κ , and η is not uniquely defined. We can use this freedom to minimize the weight of the non-commuting operator, and thus to optimize the effective classical description [7]. Following the preceding paper, we introduce a new partition for the noise term in the equation of motion, according to

$$\tilde{\eta}_b = (1+s) \eta_b, \quad (44)$$

$$\tilde{\kappa}_b = \kappa_b - s \eta_b. \quad (45)$$

For these new operators we have

$$[\chi_b(t), \tilde{\eta}_b(t)] = (1+s) [\chi_b(t), \eta_b(t)] = 0, \quad (46)$$

$$[\chi_b(t), \tilde{\kappa}_b(t)] = [\chi_b(t), \kappa_b(t)], \quad (47)$$

$$[\tilde{\eta}_b(t), \tilde{\kappa}_b(t)] = (1+s) [\eta_b(t), \kappa_b(t)]. \quad (48)$$

In order to optimize the classical description we minimize the correlations $\langle \tilde{\kappa}_b \tilde{\kappa}_b \rangle$ and $\langle \tilde{\eta}_b \tilde{\kappa}_b \rangle$. Assuming that the correlations are real we have

$$\left| \frac{\langle \tilde{\kappa}_b(t-\delta t) \tilde{\kappa}_b(t+\delta t) \rangle}{\langle \tilde{\eta}_b(t-\delta t) \tilde{\eta}_b(t+\delta t) \rangle} \right|_{k=k_0 \bar{\beta}} \ll 1. \quad (49)$$

The function that minimizes the relation between the expectation values is

$$s(t) = \frac{\langle \kappa_b \kappa_b \rangle + 1/2 \langle \kappa_b \eta_b + \eta_b \kappa_b \rangle}{\langle \eta_b \eta_b \rangle + 1/2 \langle \kappa_b \eta_b + \eta_b \kappa_b \rangle}. \quad (50)$$

This $s(t)$ also minimizes the relation (49) at first order in δt . From here we have

$$\frac{\langle \tilde{\kappa}_b \tilde{\kappa}_b \rangle}{\langle \tilde{\eta}_b \tilde{\eta}_b \rangle} = \frac{\langle \kappa_b \kappa_b \rangle \langle \eta_b \eta_b \rangle - 1/4 \langle \kappa_b \eta_b + \eta_b \kappa_b \rangle^2}{(\langle \kappa_b \kappa_b \rangle + \langle \eta_b \eta_b \rangle + \langle \kappa_b \eta_b + \eta_b \kappa_b \rangle) \langle \eta_b \eta_b \rangle}. \quad (51)$$

Furthermore, if we compute the correlation between $\tilde{\kappa}_b$ and $\tilde{\eta}_b$ with this value for s we have

$$\frac{\langle \tilde{\kappa}_b \tilde{\eta}_b \rangle}{\langle \tilde{\eta}_b \tilde{\eta}_b \rangle} = -4 \frac{\langle \tilde{\kappa}_b \tilde{\kappa}_b \rangle}{\langle \tilde{\eta}_b \tilde{\eta}_b \rangle}, \quad (52)$$

which implies that the same condition makes both of them negligible. Therefore, the condition for disregarding the contribution of $\tilde{\kappa}_b$, and hence having a valid classical description, is characterized by

$$Q \equiv \left| \frac{\langle \kappa_b \kappa_b \rangle \langle \eta_b \eta_b \rangle - 1/4 \langle \kappa_b \eta_b + \eta_b \kappa_b \rangle^2}{(\langle \kappa_b \kappa_b \rangle + \langle \eta_b \eta_b \rangle + \langle \kappa_b \eta_b + \eta_b \kappa_b \rangle) \langle \eta_b \eta_b \rangle} \right| \ll 1. \quad (53)$$

III. EXPONENTIAL INFLATIONARY SCENARIO

We will now apply the preceding approach to the exponential inflationary scenario. If the system is at a minimum V_0 of the instanton potential $V(\phi)$, the classical solution φ_{cl} is a constant field and the fluctuations become a quantum field with a mass $m^2 = d^2 V / d\varphi^2|_{\varphi_{cl}}$ and the Hubble constant is $H = \sqrt{4\pi V_0 / 3M_p^2}$. Therefore, the scale factor is $a(t) = e^{Ht}$ and the threshold parameter is given by $k_0 = \nu H e^{Ht}$, where $\nu = \sqrt{\frac{9}{4} - m^2 / H^2}$. Thus the equation of motion for the modes is

$$\ddot{\xi}_k + (k^2 e^{-2Ht} - \nu^2 H^2) \xi_k = 0, \quad (54)$$

and its general solution can be written

$$\xi_k(t) = A_1 H_\nu^{(1)} \left(\frac{k}{H} e^{-Ht} \right) + A_2 H_\nu^{(2)} \left(\frac{k}{H} e^{-Ht} \right). \quad (55)$$

We will use the boundary conditions which correspond to the Bunch-Davies [10] vacuum, leading to

$$\xi_k(t) = \frac{1}{2} \sqrt{\frac{\pi}{H}} H_\nu^{(2)} \left(\frac{k}{H} e^{-Ht} \right). \quad (56)$$

This is a complex wave function. To satisfy a classical interpretation the correlation of the noise must be real. The real part of the wave function (56) decreases exponentially with t , and in this case it is responsible for the imaginary part of the noise correlation. In our case $0 < k \leq \varepsilon k_0$, and thus we can state the condition to have a real noise correlation as $e^{-Ht} \ll H / (\varepsilon k_0)$. This implies that after a long enough time, $t \gg H^{-1} \ln(H / (\varepsilon k_0))$, the wave function $\xi_k(t)$ can be considered imaginary and the noise correlations real. The inflationary stage lasts $N_c \gtrsim 60$ number of e-folds, and thus we have a wide margin to reach real noise correlations. In this case we can use the approximate expression for the modes:

$$\xi_k(t) \simeq -\frac{i}{2} \sqrt{\frac{1}{\pi H}} \Gamma_{(\nu)} \left(\frac{k}{2H} \right)^{-\nu} e^{\nu H t}. \quad (57)$$

To be specific, we will define the coarse graining average by using a smooth approximant of the Heaviside function as our weight function:

$$G_\alpha(k) = \frac{1}{2} \left[1 - \operatorname{erf} \left(\frac{bk-1}{\alpha} \right) \right], \quad (58)$$

where α parametrizes the family of functions. When $\alpha \rightarrow 0$ we have $G_\alpha(k) \rightarrow \theta(1-bk)$. In this case, from Eqs. (25), (26) and using the relations $\dot{b}/b = H$ and $\ddot{b}/b = H^2$, we have

$$\dot{G} = -\frac{bHk}{\alpha\sqrt{\pi}} e^{-[(bk-1)/\alpha]^2}, \quad (59)$$

$$\ddot{G} = -\frac{bH^2k}{\alpha\sqrt{\pi}} \left[1 - \frac{2bk}{\alpha^2} (bk-1) \right] e^{-[(bk-1)/\alpha]^2}, \quad (60)$$

where $b \propto e^{Ht}$. Here $\bar{\beta} \approx 1$, and thus the variation of t is bounded by $\Delta t \sim \delta\beta/H$.

By replacing these expressions and the corresponding derivatives of the mode $\xi_k(t)$, defined by Eq. (57), in Eqs. (30)–(32), we can compute the correlation functions for the noises. They depend on integrals of the form $I_s = \int_0^\infty dk k^s e^{-(b^2+b'^2)/\alpha^2 [k-(b+b')/(b^2+b'^2)]^2}$, where $b = b(t)$ and $b' = b(t')$. When α is small they are given by the simple expression $I_s \approx \alpha \sqrt{\pi} / (b^2+b'^2) [(b+b')/(b^2+b'^2)]^s$, where we have neglected α^{-1} exponentially decreasing terms, and the correlation functions acquire the form

$$\langle \kappa_b(t) \kappa_b(t') \rangle = 4\nu^2 W(t, t') \quad (61)$$

$$\langle \eta_b(t) \eta_b(t') \rangle = \left(1 - \frac{4(bb')^2(b^2-b'^2)^2}{\alpha^4(b^2+b'^2)^4} \right) W(t, t') \quad (62)$$

$$\langle \kappa_b(t) \eta_b(t') \rangle = 2\nu \left(1 + \frac{2bb'}{\alpha^2} \frac{b^2-b'^2}{(b^2+b'^2)^2} \right) W(t, t') \quad (63)$$

where

$$W(t, t') = \frac{2^{2\nu-5} H^{2\nu-3} \Gamma_{(\nu)}^2 b b'}{\alpha \pi^{7/2} \sqrt{b^2+b'^2}} \left(\frac{b+b'}{b^2+b'^2} \right)^{4-2\nu} \times e^{-(b-b')^2/\alpha^2(b^2+b'^2) + \nu H(t+t')}.$$

Therefore, for $t = t'$ we have

$$\frac{\langle \kappa_b(t) \kappa_b(t) \rangle}{\langle \eta_b(t) \eta_b(t) \rangle} = 4\nu^2 = \left(9 - \frac{4m^2}{H^2} \right), \quad (64)$$

and thus it seems that only an inflaton with a mass of the order of the Planck mass, $m \approx \frac{3}{2}H$, could develop a classical regime for the coarse grained field. However, if we work with the redefined noise partition introduced in Sec. II, the parameter $s(t)$ which optimizes the classical description is in this approximation:

$$s(t) = 2\nu = \sqrt{9 - \frac{4m^2}{H^2}}. \quad (65)$$

Hence the condition (53) for neglecting the contribution of $\bar{\kappa}_b$ is automatically satisfied for every value of the inflaton mass because $Q \approx 0$. Therefore our necessary conditions to have an effective classical regime are satisfied with a noise given by $\tilde{\eta}_b = (1 + 2\nu)\eta_b$.

Here we have discussed the case of an inflaton field with a non-null mass. The zero mass case deserves a special discussion. As it is well known, in this case the modes (57) lead to infrared divergences for the correlation functions of the scalar field, but this does not affect the correlation functions of the quantum noises, because, according to Eqs. (30)–(32), they have kernels that contain derivatives of the weight function $G(k)$, which annihilate the contributions of the long-wave modes. In other words, the noises only live in the walls of the coarse-grained domains. Of course, the massless limit is not a realistic case and does not correspond to a truly inflationary process. A massless field driving inflation must have a nonlinear dynamics, and the nonlinear couplings will produce a feedback between the quantum fluctuations and the background. In this case the fluctuations can not be considered small perturbations to the classical field, which complicates the analysis in a highly non-trivial way. In general, our approach can be applied to a wide range of cases, provided that infrared divergencies do not appear [12].

IV. FINAL REMARKS

This work revises and generalizes our previous semiclassical approach to inflation, by considering a coarse-grained field with a smooth “wall” in momentum space. One of the most interesting advantages of this approach is that it allows us to study classicality conditions considering the whole quantum noise sector for the coarse-grained field, whereas this is not possible for a sharp cutoff because of the highly singular structure it produces for the quantum noise correlations. We divide this sector in two: a noise that commutes at equal times with the coarse-grained field and the remaining non-commuting noises. The classicality conditions discussed here arise from two considerations. One involves the relation between the different quantum noises, and the requirements for the noncommuting sector to be negligible. The other is related to the imaginary part of the correlation function of the quantum noise commuting with the coarse-grained field, which is proportional to the commutator of this quantum noise at different times. When the different-time commutator can be considered null the noise and its velocity commute, and thus the noise and its associated momentum can be considered as c-numbers, which may be viewed as a signal of classicality.

The redefinition of the noises allows us to obtain a more reliable condition for the non-commuting fluctuations to be

negligible. Furthermore, this can be essential in order to determine the existence of an effective classical regime, as it is clear in the massless exponential inflation model considered above. Without this redefinition we showed that the possibility of a classical regime is not evident, but once this redefinition is implemented and we turn to the relevant noises the situation changes completely. It becomes clear that at early times we do not have classicality because we cannot neglect either the non-commuting sector or the imaginary parts of the fluctuations. However, at later times, not only can we neglect the noncommuting sector at equal times, but we can also neglect the time correlations, and thus we can consider that a classical regime is achieved.

Furthermore, this analysis sheds light on the role of the shape of the wall of the coarse-grained domain. In the example discussed above, once we assume that the weight function satisfies (43), its shape does not affect the conditions to have an effective classical regime, although the amplitude and the spectral characteristics of the resulting classical noise is sensible to this shape. The amplitude depends on the α parameter in such a way that it increases when α decreases, diverging when the wall of the domain is given by a step function, i.e. when α becomes null. The correlation function (62) states that in general the noise is colored, and not white as generally assumed. Usually the definition of the coarse-grained field is considered as a mere mathematical artifact, but here we see that relevant physical features of the model depend on it. This opens a very interesting question regarding the possible physical origin of the structure of the domain characterizing the effective field.

Defining a classical behavior for an effective degree of freedom as we do in this work is significant, but this discussion does not exhaust the problem. The relevance of an effective degree of freedom is not only dictated by its classical or quantum behavior, but also by the correlations with the observables that we are able to measure. In such a sense, our analysis complements other approaches, such as theories for cosmological perturbations where not only the matter fields but also the metric are quantized [13], and theories where the classical world appears through the decoherence of relevant degrees of freedom under the influence of noise from a non-observed sector [14,11,15]. These last approaches are mainly based on the path integral techniques and Feynman-Vernon influence functionals, in which irrelevant degrees of freedom are integrated out. In the case we consider, the relevant degree of freedom involves an infinite number of modes, with time dependent weights. Work is under way dealing with the treatment of these effective degrees of freedom from the path integral point of view [16].

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